April 14, 2021 3:00 PM

Non-metrizability of UMF Homeomorphism of Manifolds

1 Topological Dynamics.

Top. Group G.

$G \curvearrowright X$ actions on compact spaces

G-Flows.

X is a flow, $Y \subseteq X$ a closed subspace s.t $G \cdot Y \subseteq Y$ (Sub flow)

If X is a flow, $\exists\, Y \subseteq X$ a minimal subflow. (Zorn's Lemma)

$\exists$ a minimal flow $M(G)$

For any other minimal flow Y, $M(G) \twoheadrightarrow Y$

Minimal flows Y are s.t there is no proper subflow.

$\leftrightarrow$ Any point $x \in Y$ $\overline{G \cdot x} = Y$

$M(G)$ is the Universal minimal flow (UMF).

Understanding $M(G)$ is good

Naively $\rightarrow$ Look at all quotients of $M(G)$ $\rightarrow$ generate all minimal flows.

Precisely $\rightarrow$ UMF is metrizable, dynamics is well behaved

(BYMT?) UMF being metrizable $\rightarrow$ $\forall\, Y$ minimal flow has a comeager orbit.

Equicontinuous UMF

Do metrizable UMF exists?

If G is locally compact (non compact) UMF is non metrizable.

If G is compact $x \in X$. $G \cdot x$ is compact

$G x = X$.

$M(\mathrm{Aut}(\mathbb{Q}, \leq)) = \{*\} \longleftrightarrow$ Ramsey's theorem.

(Coloring properties for set inclusion)

set embedding + increasing

$M(\mathrm{Aut}(G, \leq)) = \{*\} \longleftrightarrow$ Ramsey properties on graphs

Kechris - Pestov - Todorčević

Other results UMF

Pestov $M(\mathrm{Homeo}_+(S^1)) = S^1$

What of other manifolds?

Uspenskij $G \curvearrowright M(G)$ The action is NOT 3-transitive

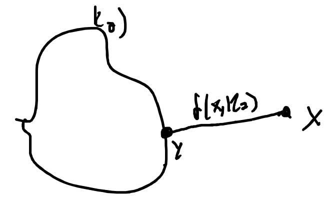
$(a, b, c) \to (a', b', c')$

Space of maximal connected chains

Let X be a compact top. space.

$VX = \{ K \subseteq X \mid K \text{ is non-empty and } \underset{\text{closed}}{\text{compact}} \}$

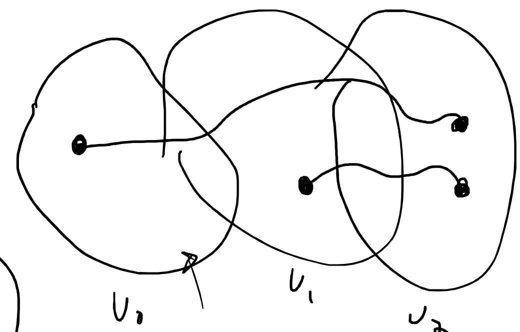
$$d(K_1,K_2) = \sup_{x\in K_1} \underbrace{\left(\inf_{y\in K_2} d(x,y)\right)}_{d(x,K_2)}$$



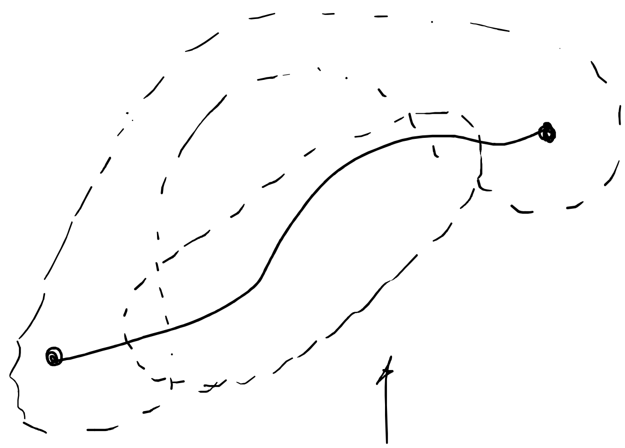
Let $U_i \underset{\text{open}}{\subseteq} X$. $0\le i\le n$

$$O(U_0,\dots,U_n) = \{K\subseteq X \mid K\subseteq \bigcup U_i \quad K\cap U_i \neq \emptyset\}$$

Example:

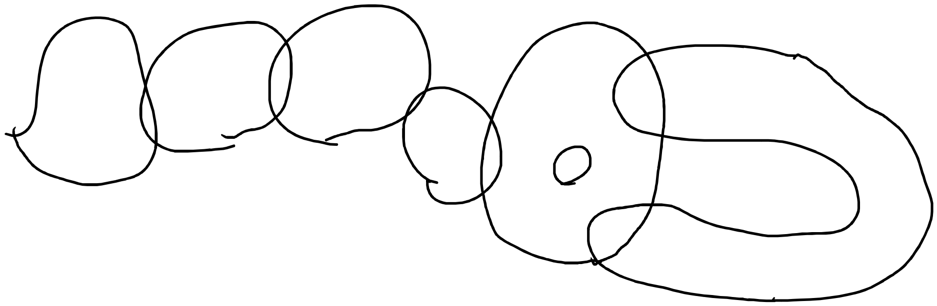


Arcs: $\varphi:[0,r]\to X$.



A tube in X is a sequence of open sets $U_0, U_1, \dots U_n$.

- U_i are connected.
- $U_i\cap U_j \neq \emptyset$ $\quad \overline{U_i}\cap\overline{U_j} \neq \emptyset$ iff $i=j\pm 1$.



$$VVX = \{ C \subseteq VX \mid C \text{ are non empty and compact} \}$$

$O(O(U_1, \ldots))$

$$C(X) = \{ C \in VVX \mid C \text{ totally ordered } \subseteq,\ C \text{ is maximal w.r.t } \subseteq,\ K \in C \text{ are all connected} \}.$$

$X \in C$

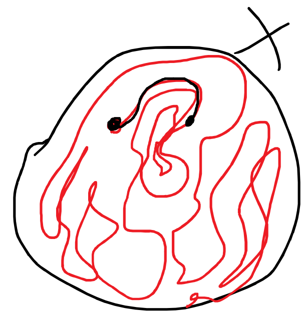
$\{x_0\} \in C$

$$C(x_0) = \{ B_r\{x_0\} : r \in [0,\infty) \}$$

- Rays.

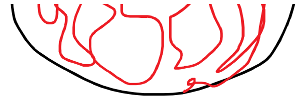
Considers injective maps of dense. $\underbrace{\phi : [0,\infty) \to X.}$

$$C_\phi = \{ \phi[0,r] : r \in [0,\infty) \} \cup \{ X \}$$



Gutman $\{C_\phi\}$ are dense in $C(X)$.

$/$ T... ...l ... bases for the topology.



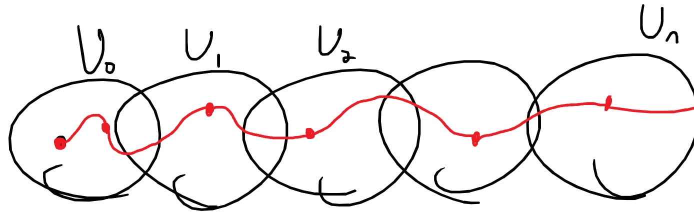
↳ Tubes induce a natural basis for the topology.

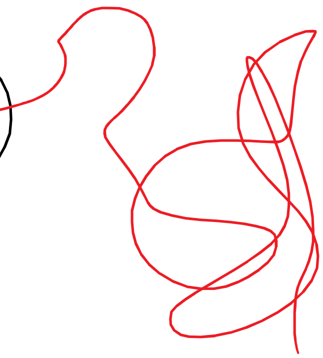
If $U_0, \dots, U_n$ is a tube

$$V(U_0, \dots, U_n) = \{ c \in C(X) \mid c \cap O(U_0, \dots, U_k) \ \forall k \le n \}.$$

$c \cap O(U_0)$

$c \cap O(U_0, U_1)$





Tubes are a basis for the topology.

Thm

$G \curvearrowright X$

$G \curvearrowright VX$

↳

$G \curvearrowright VVX$

$G \curvearrowright C(X)$ & G-action $\to$ $C(X)$ is a minimal flow

$[0,1]^{\mathbb{N}}$ $\quad d(x_i, y_i) = \sum_{n=0}^{\infty} \frac{|x_i - y_i|}{2^i}$